# PECULIARITIES OF LONGITUDINAL PROPAGATION OF MICROWAVE WITH FREQUENCY NEAR THE ELECTRON CYCLOTRON FREQUENCY IN MAGNETIZED PLASMA

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Approximate dispersion equation for quasi longitudinal propagation of slow waves with frequency near electron cyclotron frequency is derived. Perturbation technique for this equation in the wings of absorption lines gives «cold» index of refraction and finite absorption coefficient. The well known expressions for refractive index and absorption coefficient follow from solution of this equation for large enough propagation angles. Besides analytical analysis, numerical solutions of approximate dispersion equation are presented.

Nearly longitudinal propagation of wave with frequency near the electron cyclotron frequency in a magnetized plasma has a number of peculiarities due to the fact that in the cold plasma approximation refractive index for one of normal waves tends to infinity, when frequency approaches to the electron cyclotron frequency and angle between the wave vector and the magnetic field approaches zero. One of them is coalescence of upper hybrid resonance (purely longitudinal wave) with cyclotron resonance (purely transverse wave). Other is the specific behavior of cyclotron absorption coefficient for corresponding normal wave. In the dense plasma approximation

$$\frac{\omega_L^2}{\omega_B^2 \beta_T N_j \cos\theta} >> 1 \qquad (1)$$

(where $\omega_L = \left(4\pi e^2 N_e / m\right)^{1/2}$ – plasma frequency, $\omega_B = \left(eH/mc\right)$- electron cyclotron frequency, $\theta$ - angle between the magnetic field and the wave vector, $N_j$ - refractive indexes of normal waves, $\beta_T = \left(T_e / mc^2\right)^{1/2}$ - normalized thermal velocity of electrons) absorption coefficient tends to infinity when the propagation angle tends to zero [1], while for the strictly longitudinal propagation there are analytical and numerical solutions of dispersion relation which naturally result in a finite absorption coefficient [2].

In the present communication approximate dispersion relation for slow wave in a dense plasma with finite electron temperature has been derived using transition to the Stix components [3] of wave electric field and to corresponding dielectric tensor. This dispersion relation is appropriate for the frequencies corresponding to cyclotron resonance and covers the intermediate range of propagation angles from zero (strictly longitudinal propagation) to values where standard dense plasma approximation [1] is applicable. This range is of importance for the modeling of microwave power deposition into ECR discharge of axisymmetric mirror magnetic trap with the longitudinal launch of rf power.

## Dispersion equation

Let z-axis be directed along the homogeneous magnetic field $\mathbf{B_0}$. Using of Stix components $(E_\pm = (E_x \pm iE_y)/\sqrt{2}\ \ E_\| = E_z)$, for description electric field in the plane wave is equivalent to transition to rotating co-ordinates in the plane x-y, with transition matrix:

$$A = \begin{pmatrix} 1/\sqrt{2} & i/\sqrt{2} & 0 \\ 1/\sqrt{2} & -i/\sqrt{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \qquad (2)$$

«Cold» plasma dielectric tensor for Stix components doesn't contain non-diagonal elements what essentially simplifies description of longitudinally propagating normal waves.

We use the dielectric tensor for Maxwellian plasma in approximation:

$$\beta_T{}^2 |N|^2 \sin^2\theta << 1. \qquad (3)$$

This relation means that the gyro-radius of thermal electrons is much less than transverse wavelength. In this approximation dielectric tensor for Stix components takes the form:

$$\varepsilon_{ij} = \begin{pmatrix} \varepsilon_+ & 0 & 0 \\ 0 & \varepsilon_- & \sqrt{2}\xi \\ 0 & \sqrt{2}\xi & \varepsilon_\| \end{pmatrix} \qquad (4)$$

where $\varepsilon_+ = 1 - \dfrac{\omega_L{}^2}{\omega(\omega+\omega_B)}$, $\varepsilon_\| = 1 - \dfrac{\omega_L{}^2}{\omega^2}$, $\varepsilon_- = 1 + \dfrac{i\omega_L{}^2}{\omega^2 \beta_T N \cos\theta}\sqrt{\dfrac{\pi}{2}}W(Z)$,

$\xi = \dfrac{\omega_L{}^2}{2\omega\omega_B} tg\theta\left(1 + i\sqrt{\pi}W(Z)Z\right)$, $W(Z) = e^{-z^2}\left(1 + \dfrac{2i}{\sqrt{\pi}}\int_0^Z e^{\xi^2} d\xi\right)$,

$Z = \dfrac{\omega - \omega_B}{\sqrt{2}N\omega\beta_T \cos\theta}$.

Non-diagonal component $\xi$, , which is essential for absorption coefficient for the quasi-transverse propagation of waves in case of the quasi-longitudinal propagation of slow waves ($|N| >> 1$) can be neglected. After that for Stix components of electric field in the plane wave $E_\alpha \cdot \exp(-i\omega t + ik_\perp x + ik_\| z)$ the wave equation $(\text{rot}\,\text{rot}\,\mathbf{E})_\alpha - \frac{\omega^2}{c^2}\varepsilon_{\alpha,\beta}E_\beta = 0$ yields the set of homogeneous algebraic equations:

$$\begin{pmatrix} -\frac{1}{2}\left(N^2+N_\parallel{}^2\right)+\varepsilon_+ & \frac{1}{2}N_\perp{}^2 & \frac{1}{2}N_\perp N_\parallel \\ \frac{1}{2}N_\perp{}^2 & -\frac{1}{2}(N^2+N_\parallel{}^2)+\varepsilon_- & \frac{1}{2}N_\perp N_\parallel \\ \frac{1}{2}N_\perp N_\parallel & \frac{1}{2}N_\perp N_\parallel & -\frac{1}{2}N_\perp{}^2+\frac{1}{2}\varepsilon_\parallel \end{pmatrix}\begin{pmatrix} E_+ \\ E_- \\ \sqrt{2}E_\parallel \end{pmatrix} = 0\,; \quad (5)$$

from which the dispersion relation for plane waves follows:

$$\begin{aligned} &N^4(2\varepsilon_\parallel + \sin^2\theta(\varepsilon_+ + \varepsilon_- - 2\varepsilon_\parallel)) - N^2(2\varepsilon_\parallel(\varepsilon_+ + \varepsilon_-) \\ &- \sin^2\theta(\varepsilon_-(\varepsilon_\parallel - \varepsilon_+) + \varepsilon_+(\varepsilon_\parallel - \varepsilon_-)) + 2\varepsilon_\parallel\varepsilon_+\varepsilon_- = 0 \end{aligned}, \quad (6)$$

This equation agrees with standard dispersion relation for the "cold" plasma, with substitution $\varepsilon_-^c = 1 - \omega_L^2 / \omega(\omega - \omega_B)$ instead of $\varepsilon_-$. Dispersion equation (6) is transcendental ($\varepsilon_-$ is a function of complex refraction index), but formally it can be solved as biquadratic with respect to $N$. For slow waves ($|N|^2 >> 1$) corresponding solution may be obtained by omitting in (6) the last term (which does not contain $N$ in explicit form). Assuming also that inequalities $\sin\theta \approx \theta << 1$ (nearly longitudinal propagation) and $|\varepsilon_-| >> \varepsilon_+, \varepsilon_\parallel, 1$ (dense plasma, cyclotron resonance) are satisfied one can get an approximate dispersion equation:

$$N^2 \approx \frac{2\varepsilon_\parallel \varepsilon_-(N)}{2\varepsilon_\parallel + \theta^2\varepsilon_-(N)}\,. \quad (7)$$

## Limiting cases

A) For large propagation angles ($2\varepsilon_\parallel / \varepsilon_- << \theta^2 << 1$) zero order solution of Eq. (7) determines real refractive index $N^2 = 2\varepsilon_\parallel / \theta^2$ which coincides with the refractive index for the "cold" plasma in the limit $\omega \to \omega_B$. In next order the imaginary part of refractive index may be found, which determines the cyclotron absorption

$$\operatorname{Im} N = \beta_T \sqrt{\frac{2}{\pi}} \frac{2\varepsilon_\parallel{}^2\omega^2}{\theta^4\omega_L{}^2} \frac{\operatorname{Re}(W(Z))}{|W(Z)|^2} \quad (8)$$

where in argument Z the real part of refractive index from zero-order solution is involved. The identical result can be obtained from the well-known expression for cyclotron absorption coefficient of extraordinary wave at the fundamental in the limit $\theta \to 0$ [1].

B) For "small" propagation angles ($\theta^2 << 2\varepsilon_{\|}/\varepsilon_- << 1$) equation (7) becomes:

$$N^2 \approx 1 + \tilde{\varepsilon}_- - \frac{\theta^2 \tilde{\varepsilon}_-^{\,2}}{2\varepsilon_{\|}}, \; \tilde{\varepsilon}_- = i\frac{\omega_L^2}{\omega^2 N \beta_T}\sqrt{\frac{\pi}{2}} W(Z) \qquad (9)$$

Solution of equation (9) can be found using specific perturbation technique, in which zero order is numerical solution of dispersion equation for strictly longitudinal propagation of waves in a dense plasma

$$N_0^2 = \tilde{\varepsilon}_-(N_0)\,. \qquad (10)$$

Solution of this equation is the universal dependence of normalized complex refractive index $\left(\omega_B^2 \beta_T / \omega_L^2\right) N_0$ on the normalized frequency shift $(\omega - \omega_B) / \sqrt{2}\left(\omega_B \omega_L^2 \beta_T^2\right)^{1/3}$ [2]. Assuming $N = N_0 + \delta N$, where $\delta N$ is small addition $|\delta N| << |N_0|$; one can find in the first order over small parameters $\theta^2 \tilde{\varepsilon}_- / 2\varepsilon_{\|}$ and $1/\tilde{\varepsilon}_-$ :

$$\delta N \approx N_0^2\left(1 - \theta^2 N_0^4 / 2\varepsilon_{\|}\right) / \left\{\left(3 - 2Z_0^2\right) N_0^3 + iZ_0 \sqrt{2}\omega_L^2 / \omega^2 \beta_T\right\}, \qquad (11)$$

where $Z_0 = (\omega - \omega_B) / \sqrt{2} N_0 \beta_T \omega_B$ ; in receiving (11) zero order equation (10) is taken into account.

Substituting zero order solutions into the corresponding inequalities, for center of absorption line obtain two angles interval $\theta >> (\varepsilon_{\|})^{1/2}\left(\beta_T \omega_B^2 / \omega_L^2\right)^{1/3}$ and $\theta << (\varepsilon_{\|})^{1/2}\left(\beta_T \omega_B^2 / \omega_L^2\right)^{1/3}$ may be specified, where well-known expression for electron absorption coefficient in dense plasma and modified expression for nearly longitudinal propagation (10, 11) are applicable correspondingly.

C) For wings of the absorption line ($|Z| >> 1;\ \operatorname{Im} Z << \operatorname{Re} Z$) from asymptotic expansion $W(Z)$ for large argument it follows:

$$\varepsilon_- \approx \varepsilon_-^c + i\left(\omega_L^2 / \omega_B^2 N \beta_T\right) e^{-Z^2}\,, \qquad (12)$$

where $\varepsilon_-^c = 1 - \omega_L^2 / \omega(\omega - \omega_B)$ is corresponding real component of dielectric tensor for the "cold" plasma and imaginary part of $\varepsilon_-$ is much less than the real one.

In the zero approximation solution of dispersion equation (12) yields the real refractive index for the "cold" plasma

$$N_c^2 \approx \frac{2\varepsilon_{\|}\varepsilon_-^{\,c}}{2\varepsilon_{\|} + \theta^2 \varepsilon_-^{\,c}}\,, \qquad (13)$$

In the next order we obtain imaginary part of refractive index, which determine cyclotron absorption:

$$\operatorname{Im} N = \frac{1}{2}\frac{N_c^2}{\left(\varepsilon_-^c\right)^2}\frac{\omega_L^2}{\omega_B^2\beta_T}\exp\left(-Z_c^2\right), \qquad (14)$$

where $Z_c = (\omega - \omega_B)/\sqrt{2}\omega_B N_c \beta_T$ . Eqs. (13, 14) don't contain singularity at $\theta \to 0$. Applicability condition for Eq. (14)

$$|\omega - \omega_B|/\omega\beta_T >> N_c \qquad (15)$$

becomes invalid in center of absorption line and near upper hybrid resonance, where "cold" refractive index tends to infinity. For longitudinal propagation, when cyclotron and upper hybrid resonance coalesce, applicability condition for Eqs. (13, 14) is

$$(\omega_B - \omega)/\omega >> (\omega_L \beta_T/\omega)^{2/3} \qquad (16)$$

## Numerical solution of approximate dispersion equation

As it is stated above the solution of dispersion equation for strictly longitudinal propagation in a dense plasma (10) takes the form of universal dependence of normalized refractive index $\omega^2\beta_T/\omega_L^2 N_0$ on the normalized frequency shift $(\omega - \omega_B)/\sqrt{2}\left(\omega_B\omega_L^2\beta_T^2\right)^{1/3}$ . In the case of quasi-longitudinal propagation the similar procedure may be used.

Within the approximation $\varepsilon_- = \widetilde{\varepsilon}_-; \varepsilon_+ = 1 - \omega_L^2/\omega_B^2$ one can introduce normalized refractive index, frequency shift and propagation angle:

$$\widetilde{N} = N\beta_T^{1/3}(\omega_B/\omega_L)^{2/3}, \quad \xi = (\omega - \omega_B)/\sqrt{2}\beta_T^{2/3}\omega_B^{1/3}\omega_L^{2/3}$$

$$\widetilde{\theta}^2 = \theta^2\left(\omega_L^2/\beta_T\omega_B^2\right)^{2/3}\Big/\left|1 - \omega_L^2/\omega_B^2\right|; \qquad (17)$$

for these variables the dispersion equation takes the form

$$\widetilde{N}^3 = F\left(\widetilde{N}, \xi, \widetilde{\theta}\right) = i\sqrt{\frac{\pi}{2}}W\left(\frac{\xi}{\widetilde{N}}\right)\left(1 \mp \widetilde{\theta}^2\widetilde{N}^2\right), \qquad (18)$$

where the upper sign corresponds to case of plasma with density less than critical one ($\omega_L < \omega_B$), and the lower sign corresponds to case of plasma with density higher than critical one ($\omega_L > \omega_B$). Solution of this equation was found using iteration procedure with the analytical solution for zero frequency shift and zero propagation angle as zero iteration. Results of numerical solution are presented in Figs. 1, 2.

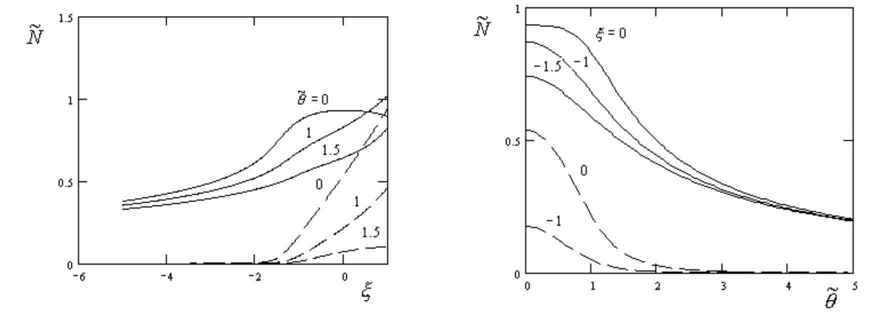


Fig.1. Dependences of normalized refractive index on normalized frequency shift and normalized propagation angle for plasma with the density less than the critical one. $\mathrm{Re}\,\tilde{N}$ - solid lines , $\mathrm{Im}\,\tilde{N}$ - dashed lines.

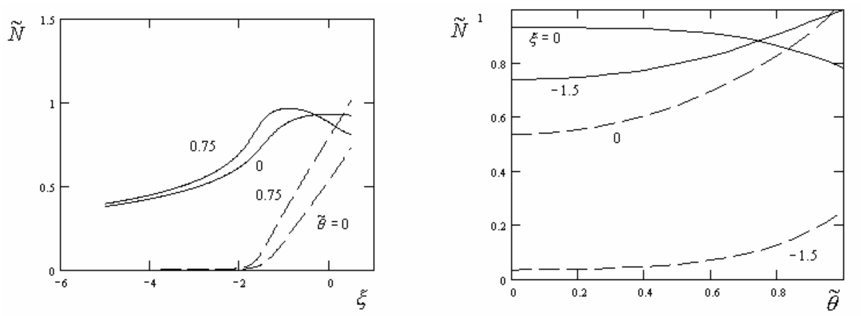


Fig.2. The same dependences for plasma with the density higher than the critical one. $\mathrm{Re}\,\tilde{N}$ - solid lines, $\mathrm{Im}\,\tilde{N}$ - dashed lines.


This work is partially supported by 03-02-17466 RFBR grant and 047.016.016 NWO-RFBR grant.


## References


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